Commutative Algebra
Fall 2013 Lecture 7

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1 A Counterexample

Last time we noted that free modules over a noncommutative ring don't necessarily have a rank, but we didn't have an example. Here's one:

Let $M = \mathbb{Z} \times \mathbb{Z} \times ...$ as a $\mathbb{Z}$ module.
Let $R = End_{\mathbb{Z}}(M)$.
Think of R as a left R-module.
Let $\phi_1(a_1, a_2, ...) = (a_1, a_3, a_5, ...)$. Let $\phi_2(a_1, a_2, ...) = (a_2, a_4, a_6, ...)$.
Let $\psi_1(a_1, a_2, ...) = (a_1, 0, a_2, 0, ...)$. Let $\psi_2(a_1, a_2, ...) = (0, a_1, 0, a_2, ...)$.
Then $(\psi_1\phi_1 + \psi_2\phi_2)(a_1, a_2, ...) = (a_1, a_2, ...)$ so $\psi_1\phi_1 + \psi_2\phi_2 = 1$.
Thus ϕ_1 and ϕ_2 generate R.
$\phi_1\psi_1 = 1, \phi_2\psi_2 = 1, \phi_1\psi_2 = 0\phi_2\psi_1 = 0$.
So if $\alpha_1\phi_1 + \alpha_2\phi_2 = 0$ then $0 = \alpha_1\phi_1\psi_1 + \alpha_2\phi_2\psi_2 = \alpha_1$.
Similarly, $0 = \alpha_1\phi_1\psi_2 + \alpha_2\phi_2\psi_2 = \alpha_2$. So $\{\phi_1, \phi_2\}$ is a base for R.
$R^2 \to R$
$(\alpha_1, \alpha_2) \mapsto \alpha_1\phi_1 + \alpha_2\phi_2$.

The same calculations show that this is an isomorphism, so rank is NOT well defined for R modules.

2 Finitely Generated Modules Over PIDs

First let's talk about torsion:

Definition: Let R be a commutative integral domain and M an R-module. Then $a \in M$ is a torsion element if $Ann_R(a) \neq 0$. Let $tor(M) = \{a \in M :$ a is a torsion element$\}$.

Note $tor(M)$ is a submodule of M:

If $a_1, a_2 \in tor(M)$ say $r_1a_1 = 0, r_2a_2 = 0$, $r_1 \neq r_2$, then $r_1r_2(a_1 + a_2) = 0$ so $a_1 + a_2 \in tor(M)$ and for any $r \in R$ $r_1ra_1 = r(r_1a_1) = 0$ so $ra_1 \in tor(M)$.

Proposition: (Basic Torsion Facts)
Let R be a commutative integral domain.

1. If $f : M \to N$, then $f(tor(M)) \subseteq tor(N)$.
2. $tor(M_1 \oplus M_2) = torM_1 \oplus torM_2$.
3. $R^{(n)}$ is torsion free.

Proof:

1. Take $a \in tor(M)$ with $ra = 0, r \neq 0$. Then $f(ra) = rf(a)$, so $f(a) \in tor(N)$.
2. Let ν_1, ν_2 be the canonical maps for $M_1 \oplus M_2$, by 1 $\nu_i(tor(M_i)) \subseteq tor(M_1 \oplus M_2)$ so $torM_1 \oplus tor(M_2) \subseteq tor(M_1 \oplus N_2)$ and if $(a_1, a_2) \in tor(M_1 \oplus M_2)$ say $r(a_1, a_2) = 0, r \neq 0$ then $ra_1 = 0, ra_2 = 0$.
3. R is torsion free over itself since it has no zero divisors. Then apply 2.

Proposition: Let R be a PID. Any submodule of a free module of rank n is free of rank at most n.

Proof: By induction on n.

Let M be the module. If $n = 1$ then $R = M$ the submodules of R are its left ideals, which are of the form Rr for some $r \in R$ and $Ann_R(r) = 0$ since R has no zero divisors so $\{r\}$ is a base for Rr if $r \neq 0$, otherwise $Rr = 0$.

Suppose M is a submodule of $R^{(n)}$. Let $\{e_i\}$ be the standard base.
Let $\phi : R^{(n)} \to R^{(n-1)}$.

$e_i \mapsto e_i \ (i < n)$

$e_n \mapsto 0$.

By induction, $\phi(M)$ is free of rank $\leq n - 1$. $Ker\phi = Re_n$. So MRe_n is free of rank ≤ 1. So by the proposition that ranks add, the $rank(M) \leq n$.

Proposition: a finitely generated torsion free module over a PID is free.

Proof: Let $\{x_1, ...x_n\}$ span M with M torsion free. Let $\{v_1, ...v_l\}$ be a maximal subset of M with $\Sigma r_i v_i = 0 \Rightarrow r_i = 0$.

For $x_i \notin \{v_1, ...v_l\}$ by maximality $\exists s_i \in R, s_i \neq 0$, such that $s_i x_i + b_{i1} + ... + b_{il} v_l = 0$.

Let s be the product of the s_i. Then $sM \subseteq N$ where N is the module spanned by $\{v_1, ...v_l\}$ then the map $M \to N$ defined by $x \mapsto sx$ is injective since M is torsion free. So sM is free as it is a submodule of a free module and $M \cong sM$ so M is free.

Proposition: Let R be a PID, and M be a finitely generated R-module.
Then $M \cong tor(M) \oplus M/tor(M)$. $M/tor(M)$ is finitely generated and torsion free, and so $M/tor(M) \cong R^{(r)}$ for some r.

Proof: First let's check that $M/tor(M)$ is torsion free. Take $x \in tor(M)$ and suppose it has a non-zero annihilator, so $\exists r \in R, r(x + tor(M)) = tor(M), r \neq 0$.

So $rx \in tor(M)$ so $\exists s \in R, s \neq 0$ such that $srx = 0$ but $sr \neq 0$ so $x \in tor(M)$. Next note that $M/tor(M)$ is finitely generated so we can use the previous result.

Consider $0 \to tor(M) \to M/tor(M) \to 0$. This is exact (with π from M to $M/tor(M)$. We want it to be split. $M/tor(M)$ is free, let $\{x_1, ...x_n\}$ be a base. Pick $a_i \in M$.

$\pi(a_i) = x_i$.

Then define

$g : M/tor(M) \to M$

$x_i \mapsto a_i$.

This exists and gives the splitting, so we are done.

Recall in a PID, every irreducible is prime.

Definition: Let p be a prime of R. Then $M_p = \{m \in tor(M) : \exists i : p^i m = 0\}$ is a p-primary module.

Proposition: M_p is a submodule of M.

Proof: Take $a_1, a_2 \in M_p$. Then $p^i a_1 = 0, p^j a_2 = 0$, so $p^{i+j}(a_1 + a_2) = 0$ and $p^i r a_1 = r p^i a_1 = 0$.

Proposition: $tor(M) = \oplus M_p$ where the sum runs over a finite set of primes of R.

Proof: Choose $\{p_i\}_{i \in I}$ such that every prime of R can be uniquely written as up_i for some unit u and some i. Consider $\phi : \oplus M_{p_i} \to tor(M)$
$(m_i) \to zm_i$

Check that ϕ is an isomorphism. Take $x \in tor(M), x \neq 0$, then $Ann_R(x)$ is a nonzero ideal of R so it is principal, say $Ra = Ann_R(x)$ so write $a = u\pi p_i^{n_i}, u$ unit.

Consider the elements $a/p_i^{u_i}$, the gcd of these elements is 1 so $\exists r_i$ such that $1 = \sigma r_i a/p_i^{n_i}$. So $x = \sigma r_i a/p_i^{n_i}$ x. Further, $p_i^{n_i} x_i = ax = 0$, so $x_i \in M_{p_i}$, so x = $\phi((r_i x_i))$, thus ϕ is onto.

Next, suppose $(x_i) \in Ker\phi$. Then $\Sigma x_i = 0$. Suppose $p_i^{n_i} x_i = 0$. WLOG, say i runs from 1 to n. Then $p_2^{n_2} ... p_n^{x_n}$ annihilates $x_2, ...x_k$ so $p_2^{n_2} ... p_k^{n_k} \in Ann_R(x_1)$, but x_1 is from the p_1-primary part so $Ann_R(x_1) = Rp_1^{n_1}$ which is impossible by unique factorization.

Thus ϕ is an isomorphism.

Finally, since M is finitely generated, $M \cong (\oplus M p_i) \oplus M/tor(M)$. So each generator of M uses only finitely many summands, thus all together they use only finitely many summands, so the whole sum only involves finitely many pieces. It remains to check that each M_p is a direct sum of cyclic modules.

Definition: A submodule N of M is pure if whenever $ax \in N, a|inR, x \in M$

then $\exists z \in N$ such that $az = ax$.

<u>Lemma:</u> If $P = Rx_0$ is a pure cyclic submodule of a finitely generated module N, and N/P is a direct sum of cyclic modules, then $N = N/P \oplus P$.

<u>Proof:</u> Let $x_i + P$ be a set of generators for the summands of N/P (finitely many since N is finitely generated, say $i = 1, .., k$).
Let a_i generate $Ann_R(x_i + P)$, so $a_i x_i \in P \forall i$. Since P is pure, $\exists z_i \in P$ such that $a_i z_i = a_i x_i$. Let $y_i = x_i - z_i$. Then $a_i y_i = 0$ so $a_i \in Ann_R(y_i)$ and if $a \in Ann_(y_i)$ then $ax_i = az_i$ so $a \in Ann_R(x_i + P)$ so $Ann_r(y_i) = Ann_R(x_i + P) = Ra_i$. So by the first isomorphism theorem $Ry_i = R/Ann_R(y_i) = R/Ra_i = R(x_i + P)$.

So it suffices to prove that $M = Rx_0 \oplus (\oplus Ry_i)$.

So take $m \in M$, $m + P = \sigma r_i(x_i + P) = \sigma r_i(y_i + P)$. Then $m - \sigma r_i y_i \in P_k$.

Thus $m \in P \oplus (\oplus P y_i)$.

For directness, note, say $r_0 x_0 + r_1 y_1 + ... r_k y_k = 0$, then $r_1(y_1 + P) + ... + r_k(y_k + P) \in P$, so by directness of N/P, $r_1 = r_2 = ... = r_k = 0$.
So $r_0 x_0 = 0$, so $r_0 = 0$. Thus sum is direct.